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NONLINEAR HYDRODYNAMICS

We now come to a very sophisticated method for calculating the stability and pulsations of stars which make contact with actual observations of the stellar behavior. Hydrodynamic calculations are very simple in principle. Conservation of mass can be accounted for by having mass shells that are fixed with their mass for all time. Motions of these shells can be calculated by taking the difference between the external force of gravity and that from the local pressure gradient. The conservation of energy can be coupled to this momentum conservation equation to give the current temperatures, densities, pressures, and opacities at the shell centers, as well as the positions, velocities, and accelerations of the mass shell interfaces. Energy flow across these interfaces can be calculated from the current conditions, and this energy is partitioned between internal energy and the work done on or by the mass shell. We will discuss here only the purely radial case for hydrodynamics because it is very useful for stellar pulsation studies.

The one-dimensional initial value computer programs for calculating stellar pulsations are few in number. Christy (1964) published the first description of a stellar hydrodynamics code. It was based on techniques developed at Los Alamos in earlier days. We also used the same general methods and published a description in a paper by Cox, Brownlee, and Eilers (1966). There have been several programs since, and their descriptions are given by Stobie (1969), Baker and von Sengbusch (1969), Bendt and Davis (1971), Stellingwerf (1974), Wood (1974), Spangengerg (1975), Karp (1975), Castor, Davis, and Davison (1977), and Vemury and Stothers (1978).

All these programs were written for studies of Cepheids and RR Lyrae variables except for the Wood and Karp programs. The Wood program has been used only for calculations of Mira variables, and the Karp program is actually the nova code developed by Kutter and Sparks (1972). Mira pulsations have also been studied with the Christy code (Keeley 1970) and our Los Alamos DYNSTAR code. The Christy code was further used by Trimble (1972) for helium star pulsations, and the Stellingwerf code was used some for δ Scuti pulsation calculations. Both the Stellingwerf and the Los Alamos codes have features that can seek

strictly periodic solutions by a relaxation technique, but this valuable procedure for Cepheids will not be discussed here because it does not work for the very slowly decaying pulsations of the upper main sequence stars.

Appenzeller (1970), Ziebarth (1970), and Talbot (1971) have written computer codes for the calculation of pulsations of very massive upper main sequence stars. Later Papaloizou (1973), using essentially the Christy techniques, has calculated the hydrodynamics of stars masses between 70 and 210 solar masses. These calculations were not very detailed and were not very definitive in their predictions of possible pulsationally induced mass loss. Nowadays the facts seem to be that stellar winds produce so much mass loss that even the lowest mass considered by these authors, sixty solar masses, have very short lives. It is still possible, however, that pulsation is responsible for the very high mass loss rates seen in stars such as the Wolf-Rayet stars.

The Appenzeller, Kutter-Sparks, Stellingwerf, Castor, and our DYNSTAR codes are completely implicit. That means that the equations are time centered at least halfway across the time step, and therefore the solution necessarily involves iterations for quantities at the advanced time. The Courant time step limit (see Richtmyer and Morton 1967) is not necessary for these codes, and if the accuracy allows a larger time step, it can be used. The energy equation is solved implicitly, however, in all the above described numerical programs.

The current version of the DYNSTAR program will be described here. As one might expect, all these above-mentioned programs have evolved or died, and the published descriptions are generally not accurate. The current DYNSTAR program, made completely implicit in 1975 is actually a version of an unpublished fortran code written by Carl Hansen at Boulder, based on our published concepts, and some of his, and finally shipped back to Los Alamos!

The starting configuration of the Lagrangian mass shells is that given by the envelope model code described in the fifth lecture. The figure there gives the zone numbering, but that was to accord with the Castor (1971) system for the linear pulsation eigensolutions. For DYNSTAR we relabel the zone interfaces with a number one less than given there. This means that the outer mass shell center is named IR and so is the outer (surface) interface. The central ball radius needs no

number because it remains fixed in space for all time and produces a fixed luminosity.

Figure 1 gives the two basic equations for DYNSTAR. These are the momentum and the energy equations. The equation for mass conservation in our Lagrangian system is only an auxiliary one which gives the specific volume of each mass zone at the beginning of each timestep and during the implicit method iterations for the end of the timestep. The hydrodynamic solution is really just a series of models spaced a timestep apart, where this timestep is as large as possible to retain accuracy of the solution.

Values of all the variables are known at time n. For the needed values at the next time n+1, there is an extrapolation of the quantities and then a set of a few iterations to improve these values so that the equations are accurately satisfied. Because we have what is called a completely implicit method, there is no mathematical (Courant) stability limitation on the timestep as there is when the hydrodynamic behavior is calculated explicitly.

1-D LAGRANGIAN HYDRODYNAMICS-TWO VARIABLES
MASS, MOMENTUM, AND ENERGY CONSERVATION

$$\Delta t = t^{n+1} - t^{n};\ \text{interface } I;\ \text{mass zone } I;\ 1 \leq I \leq IR$$

$$V = \frac{4\pi}{3M_I}\left(r_I^3 - r_{I-1}^3\right)$$

$$M = \ddot{r}_I^{n+\frac{1}{2}} + 2\dot{r}_I^{n}/\Delta t - 2\Delta\, r_I^{n+\frac{1}{2}}/(\Delta t)^2 = 0$$

$$E = \Delta Q_I^{n+\frac{1}{2}} - \Delta E_I^{n+\frac{1}{2}} - p_I^{n+\frac{1}{2}}\,\Delta V_I^{n+1} = 0$$

Figure 1. The basic equations of mass, momentum, and energy conservation are integrated over timesteps for all the Lagrangian mass zones.

Our so-called momentum equation is really an expression relating the interface positions, their velocities and accelerations. The basic quantity is the acceleration which is time centered by averaging the pressure gradient and the force due to the gravity of the internal mass over the timestep. The velocity at time n+1 is that at time n plus the acceleration at the mean time times the timestep. Likewise, the

difference in the interface radius between time n+1 and time n, that is, δr centered at time $n+\frac{1}{2}$, is the velocity centered at time $n+\frac{1}{2}$ times the timestep. This velocity at the midtime is taken as the mean of its value at time n and n+1. Combining all these things results in what we call the momentum equation, and it is iterated together with all the provisional quantities such as temperature, specific volume, pressure, etc at time n+1 until the equation equals zero.

Our energy equation is the first law of thermodynamics. Energy flowing into or out of the mass shell by radiation, conduction, or convection plus the energy introduced into the mass shell by nuclear burning is partitioned between internal energy and work according to this equation. Time centering of all these terms means that the variables must be guessed and iteratively improved for time n+1, just as for the momentum equation. Convergence for both of the basic equations is reached when the equations attain values small compared to their largest terms.

The momentum equation that describes the zone interface acceleration is given in Figure 2. Fraley (1968) has shown that to conserve energy exactly through the entire envelope model, it is necessary to define the mean square of the interface radius in the way given which involves both times n and n+1. Also it is necessary to use the acceleration due to gravity at a mean radius as given again involving the radius at both ends of the timestep. The figure also shows some detail about the nonadiabatic term of the energy equation. Luminosity by radiation and conduction is calculated with the normal Rosseland diffusion equation where the κ includes both radiative and conductive contributions.

These two equations of our system involve many variables, and we choose two of them (T,r) for each mass shell as our unknowns. All other variables, such as the density, the pressure, the opacity, the luminosity, etc. can be calculated with the shell mass and these two dependent variables known. Iterations involve the corrections to T and r for each shell so that convergence, that is, consistency of the two basic equations, is obtained over the timestep.

Note that there are many other possible procedures of getting the hydrodynamic solutions such as using more equations and more variables. The Stellingwerf method has three equations (velocity, acceleration, and

ID LAGRANGIAN HYDRODYNAMICS-TWO VARIABLES
ACCELERATION AND NONADIABATIC TERMS

$$\ddot{r}_I^{n+\frac{1}{2}} = = 4\pi\, \bar{r}_I^{-2} \left(\frac{dP}{dM}\right)_I^{n+\frac{1}{2}} - g_I^{n+\frac{1}{2}}$$

$$\bar{r}_I^{-2} = 1/3({}^{n}r_I^2 + {}^{n}r_I\, {}^{n+1}r_I + {}^{n+1}r_I^2)$$

$$g_I = \frac{GM(r)}{{}^{n}r_I\, {}^{n+1}r_I}$$

$$\Delta Q = L_{in} - L_{out}$$

$$L = 4\pi\, r_I^2 \frac{ac\, V_I}{3\kappa_I} \left(\frac{dT^4}{dr}\right)_I$$

Figure 2. The acceleration and nonadiabatic terms are given for the numerical hydrodynamics calculation.

energy) and three unknowns (T, r, and the velocity u). The Kutter-Sparks system uses five. These are the conservation of mass, momentum, energy, the total of the radiative, conductive, and convective luminosities, and the definition of velocity. The unknowns here are the T, r, V, the total luminosity and the velocity. Appenzeller (1970) uses the same equations but assigns the unknowns as T, r, pressure, the total luminosity and the velocity. The backward time differencing (not time-step centered) that he used was very dissapative, however, even though quite stable. The Castor variable zone mass (mixed Lagrangian and Eulerian) DYN code uses an explicitly determined mass together with an implicit solution for T, r, and u using the equations of mass, momentum, and energy.

I am not sure which system is the best. In these completely implicit codes a matrix solution is made for each iteration as we will soon see. If there are more equations, there are more rows of the matrix and more components of the correction vector. This certainly means more matrix operations. In our case, if the number of equations is reduced to only two for each zone, then the matrix is more simple. Nevertheless, the saving in matrix operations is used up in the process of developing all the terms of the derivatives as elements of our more elaborate two equations.

For our two basic equations we need to assign boundary conditions at the top and bottom of the envelope. The central ball has a fixed radius and a fixed core luminosity which is the surface luminosity minus any thermonuclear energy sources or neutrino losses in all the envelope zones. At the surface the acceleration of the interface is taken as

$$\ddot{r}_{IR} = f_A \frac{4\pi \bar{r}_{IR}^{-2}}{M_{IR}} (0 - P_{IR}) = g_{IR} \qquad (1)$$

There the luminosity is calculated by an approximate radiation transport solution for the effective temperature as a function of the temperature at the center of the last zone and the optical depth at that point.

$$T_{IR}^4 = 3/4\ T_e\ (\tau_{IR} \cdot 2/3) \qquad (2)$$

With this effective temperature, the luminosity can be calculated from the normal black body luminosity formula

$$L_{IR} = 4\pi R_p^2 \sigma T_e^4 \qquad (3)$$

As we have indicated before, our implicit method requires iterations to converge on values of the two dependent variables at the advanced time. A Newton-Raphson procedure is followed as indicated in Figure 3. The matrix we deal with consists of many derivatives of which there are 2 with respect to T and 3 with respect to r in the momentum equation, and there are 3 with respect to T and 4 with respect to r in the energy equation. Thus the matrix is 7 diagonal.

The notation used in the actual program is also given in Figure 3. Each of these derivatives is calculated each iteration k from sometimes rather complicated formulas. These derivatives are obtained analytically except for the derivative of the convective luminosity with respect to T and r. These latter derivatives are done numerically by applying an increment of typically 10^{-7} to T and r separately and then recalculating the convective luminosity. The difference in the luminosities divided by the difference in the variable (T or r) gives a partial derivative which is accurate enough to guide the iterations to converg ace.

The Newton-Raphson method is applied by putting the negative of the energy and momentum equations, called respectively ZC and α, as

ID LAGRANGIAN HYDRODYNAMICS - TWO VARIABLES
NEWTON-RAPHSON ITERATIONS

$$\alpha_I = M_I^k + \sum_{i=I}^{I+1} \left(\frac{\partial M}{\partial T}\right)_i^k \delta T_i + \sum_{i=I-1}^{I+1} \left(\frac{\partial M}{\partial r}\right)_i^k \delta r_i = 0$$

$$ZC_I = E_I^k + \sum_{i=I-1}^{I+1} \left(\frac{\partial E}{\partial T}\right)_i^k \delta T_i + \sum_{i=I-2}^{I+1} \left(\frac{\partial E}{\partial r}\right)_i^k \delta r_i = 0$$

Band matrix solution for known M, E and derivatives at any iteration k gives the vector δT_1, δr_1, δT_2, δr_2, ... → 0.

$$\beta_I = \frac{\partial M_I}{\partial r_{I-1}} \qquad AC_I = \frac{\partial E_I}{\partial T_{I-1}}$$

$$\delta_I = \frac{\partial M_I}{\partial r_{I+1}} \qquad BC_I = \frac{\partial E_I}{\partial T_I}$$

$$\delta_I = \frac{\partial M_I}{\partial r_I} \qquad CC_I = \frac{\partial E_I}{\partial T_{I+1}}$$

$$\eta_I = \frac{\partial M_I}{\partial T_I} \qquad DC = \frac{\partial E_I}{\partial r_{I-1}}$$

$$\phi_I = \frac{\partial M_I}{\partial T_{I+1}} \qquad EC = \frac{\partial E_I}{\partial r_{I+1}}$$

$$FC = \frac{\partial E_I}{\partial r_I}$$

$$GC = \frac{\partial E_I}{\partial r_{I-2}}$$

Figure 3. Details of the Newton-Raphson method are given for the implicit solution.

components of a right-hand side vector. The linear equations are solved for a correction vector consisting of increments for the T, r pairs. This matrix is portrayed in Figure 4. Since the matrix elements are changing each iteration, considerable calculation is involved, but usually only 4-5 iterations are needed to get convergence. If the time step can be ten or more times that given by the Courant limit, the use of this implicit iteration procedure is warranted.

We need now to discuss the guesses that are made for the T, r values for the advanced time n+1. In Figure 3, the momentum and energy equations, that is, α and ZC for each zone, were time centered between step n and n+1. A simple example of this is that in the energy equation the dE is just the internal energy at time n+1 minus its value at time n. Consider now the momentum and energy equations at time n+½, in order to make our extrapolation or guesses for time n+1. A Taylor expansion of the two equations would result in almost exactly the two expressions given at the top of Figure 3, and they both equal zero because these two equations are here assumed to be zero at the mean time. The Taylor expansion, however, involves only quantities at time n and none at time n+1. These equations at time n and their various derivatives are somewhat different than they were in the implicit method iterations which involve time centered quantities. If the extrapolation is made to the mean time, the derivative terms should be multiplied by ½. Solution for the correction vector, as described before, now allows them to be applied to give a very good approximation to the configuration at time n+1, and a good starting point for the iterations. We call this Taylor expansion the explicit P step and the iterations as the F step of the time step.

In DYNSTAR, all the time centered quantities are exactly half of their time n value and half of the time n+1 value except for the luminosities. Following Stellingwerf (1975), we give the n+1 luminosities a weight of 2/3 and those at time n only 1/3. Presumably this makes the integrations more stable without much loss in accuracy. With our centering at exactly at the time midpoint, the procedure is marginally stable, and sometimes we have convergence trouble for large timesteps.

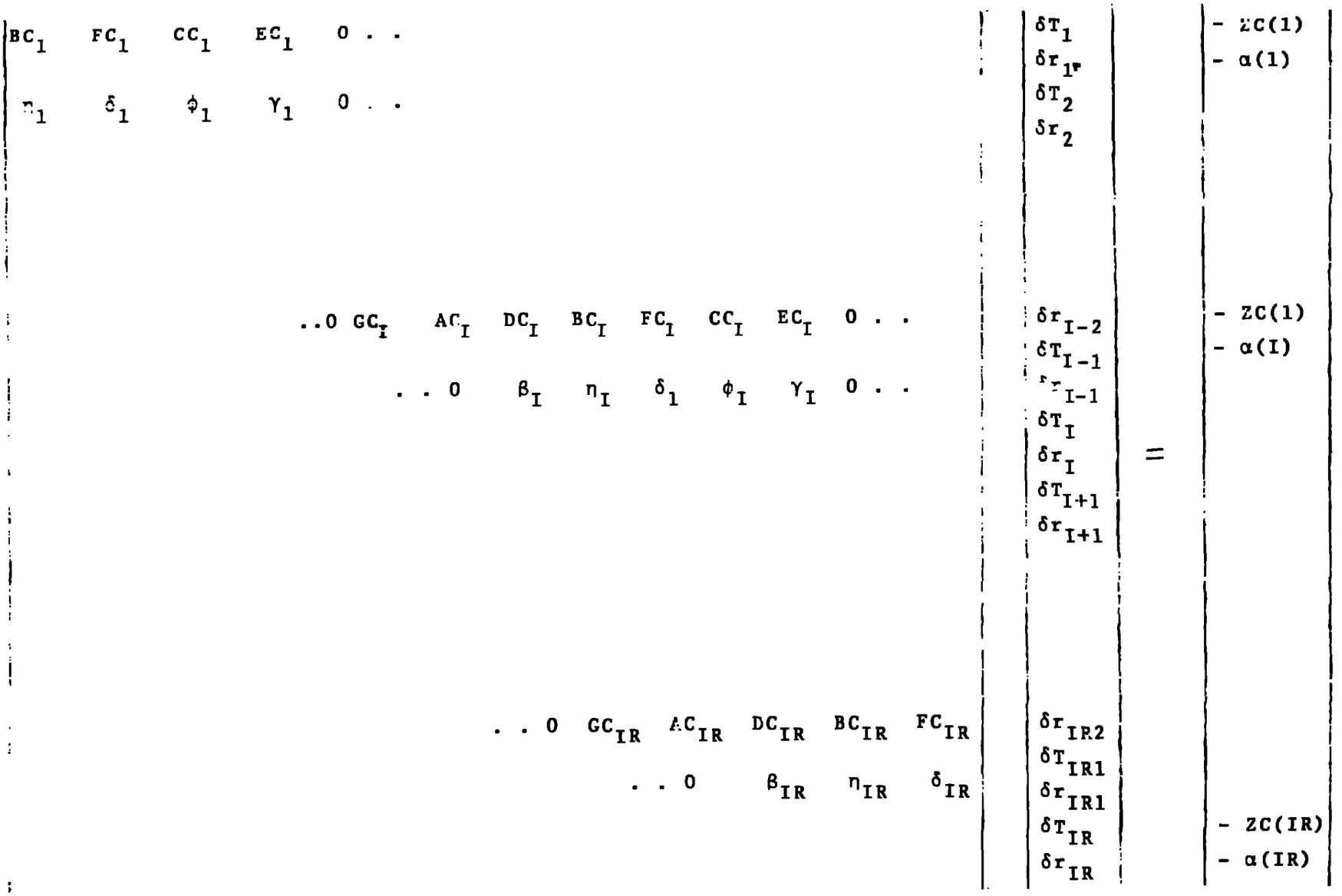

Figure 4. The structure of the 7 diagonal matrix for the solution for the correction vector is displayed using the quantities defined in the previcus figure.

A very important thing to do when using the Newton-Raphson method is to be sure that the largest increment in the T, r correction vector is not so large that the solution iterations are upset and nonconvergent. In the normal case, the maximum, wherever it is in the zones, can be as large as 30%, but sometimes a much smaller correction needs to be applied so that the equations do not stray out of their radius of convergence.

Time dependent convection is necessary to give realistic hydrodynamic solutions in most cases. Figure 5 presents the mixing length equations with a modification for the convective velocity variation. As used today, the increase of the convective velocity each time step is 0.1 or less times the velocity of an element (Δv) after it has travelled a mixing length. The decrease is given by the fomula for the lag, and if the gradient becomes subadiabatic, the drag is multiplied by 1000. These parameters do not matter much for upper main sequence stars, because the surface convection zone is very weak and the core convection timescale is perhaps 100 times the pulsation period.

All of the usual hydrodynamic codes, for pulsation or not, need an artificial viscosity to smooth out strong disturbances such as shock waves. The use of the viscosity is simple. The pressure which consists of gas and radiation terms as well as perhaps the convective turbulent pressure term, is further increased by the turbulent eddy and artificial viscosity terms. This pressure, used both in the momentum and energy equations, is given as

$$q_I = qa_I + qt_I$$

$$qa_I = \frac{FA}{V_I}\left[\dot{r}_I^{n+1} - (r_{I-1}/r_I)^2\,\dot{r}_{I-1} + \theta VS_I\right]^2 \qquad (4)$$

$$\text{if } [\ ] \text{ term} < 0$$
$$qa_I = 0 \text{ otherwise} \qquad (5)$$

$$qt_I = \frac{FT}{V}\, v_c\, \Lambda_c\, \frac{d\dot{r}}{dr} \qquad (6)$$

v_c and Λ_c = qllp from time dependent mixing length theory

TIME DEPENDENT CONVECTION

$$\text{If } v_c \geq v_c^n \qquad v_c^{n+1} = v_c^n + \Delta v$$

$$\Delta v = \frac{\Delta\rho}{\rho}\, g\, \frac{2\Lambda}{r_{i+1} - r_{i-1}}\, \Delta t\, AF \geq 0$$

$$\frac{\Delta\rho}{\rho} = Q\, \frac{\Delta T}{T}$$

$$\text{if } v_c < v_c^n$$

$$v_c^{n+1} = v_c^n + \tau(v_c - v_c^n) \geq 0$$

$$\tau = \frac{v_c^n + v_c^{n+1}}{2}\, \frac{\Delta t}{\Lambda}\, LF$$

$$L_c \sim v_c^3$$

$$LF = 1$$

$$\text{IF } v_c < 0 \qquad \frac{\Delta\rho}{\rho} < 0 \quad \text{use } \Delta v < 0$$

$$\text{with} \qquad AF = 1000$$

Figure 5. The equations for the time dependent convection are based on a lag of the convective element velocity in mixing length theory.

We sometimes use a linear artificial viscosity where the square bracket term is not squared, the threshold θ is set to 0, and a factor of the sound velocity is added.

Material properties are needed to carry out calculations. We have already discussed the opacities, and the equation of state needed to calculate them is also needed to calculate the pressure and energy for the mass shells in the hydrodynamics integrations. Both tabular and analytic formulas are used, the latter when it seems very necessary to operate with smooth opacity and equation of state derivatives as for

convection or just for ease of the iteration convergence. The standard thermonuclear reaction rates of Fowler, Caughlan, and Zimmerman (1975) and screening factors discussed by Reeves (1965) are used for any energy source present.

Opacities vary rapidly with temperature and they are known only at the center of the mass shells in our hydrodynamic calculations. Yet, the radiative luminosity passing from shell to shell is space centered at the shell interfaces. Simple averaging of the opacities in adjacent zones is not adaquate especially if the number of zones is limited to typically 50-100. Christy (1967) and Stellingwerf (1975) have proposed averaging procedures. Without giving the procedure here, we merely say that we use the Stellingwerf method.

Selection of the time step is crucial for the calculations. If the step is too small, results will be slow in coming. If the time step is too large, the iterations will not converge. We have adopted a time step selection procedure that retains the previous time step if the number of iterations for the last time step was reasonable, say six to ten. For a larger number of iterations, the problem is having trouble and the time step is cut by 15%. If convergence is very easy (in less than six iterations), multiplication of the timestep by 1.15 is made for the next one.

The typical way a pulsation solution is started is to use the actual eigenvector from the linear nonadiabatic solution. This eigenvector is rescaled from its usual normalization with a surface amplitude of 100%. We start from a hydrostatic and thermal equilibrium model and apply the radius eigenvector as the outward velocity structure. To get the T, r configuration that obtains at this midpulsation position we look at the imaginary part of the T, r vector, which pertains to a time a quarter of a cycle earlier than the maximum radius time. Incrementing the temperatures, radii, and velocities now puts the model out of thermal and hydrostatic balance and the integration (with a not too large timestep) begins.

As an example of the use of the hydrodynamic code, we here give results of a calculation for the β Cephei variable α Virginis. The mass of the model is 11.5 solar mass, the effective temperature is 26,000 K, and the luminosity is 6.5×10^{37} ergs/s. This is exactly the model discussed in lecture 1 for its linear theory pulsation solution.

This linear solution has been scaled here to a photospheric peak to peak radial velocity of 24 km/s to allow for the limb darkening which affects the 17 km/s observed amplitude.

Figure 6 shows the radial velocity variations for about 10 periods. All cycles do not repeat exactly in this calculation. Figure 7 is the radius variation from the hydrodynamic integration, and a peak-to-peak variation of about 1% is seen. Since all the interior variations are even smaller, and the decay e-folding rate for this fundamental mode is only about one part in 10,000, it is reasonable to expect that this model is behaving very closely to the result from linear adiabatic theory.

Bolometric magnitude variations are displayed in Figure 8. Here the amplitude peak-to-peak is 0.19 magnitude which corresponds to a much smaller amplitude in the visual band. The surface effective temperature variations of 650 K above and below the mean of 26,000 K, given in Figure 9, produce a varying bolometric correction of close to 0.11 magnitude (Code et al.1975) with the maximum correction occuring at the hottest phase which is also the most luminious phase at minimum radius. These corrections are -2.63 at 26,650 K and -2.52 at 25,350 K. Subtraction of this correction difference then produces somewhat more than the actual few hundredths of a magnitude observed variation in the V filter.

The kinetic energy of the motions varies as shown in Figure 10. Twice each cycle, at maximum infall and maximum expansion velocity, the kinetic energy reaches a maximum. For other stars that are not so adiabatic, there are phase differences between the maximum velocity at different levels in the star, and the peaks and valleys of this kinetic energy variation are not so separated as in our case where there is a range of three or more powers of ten.

We believe that the supergiant pulsations are all nonradial, because we will show how very rapid the decay of a radial hydrodynamic disturbance is in our final lecture.

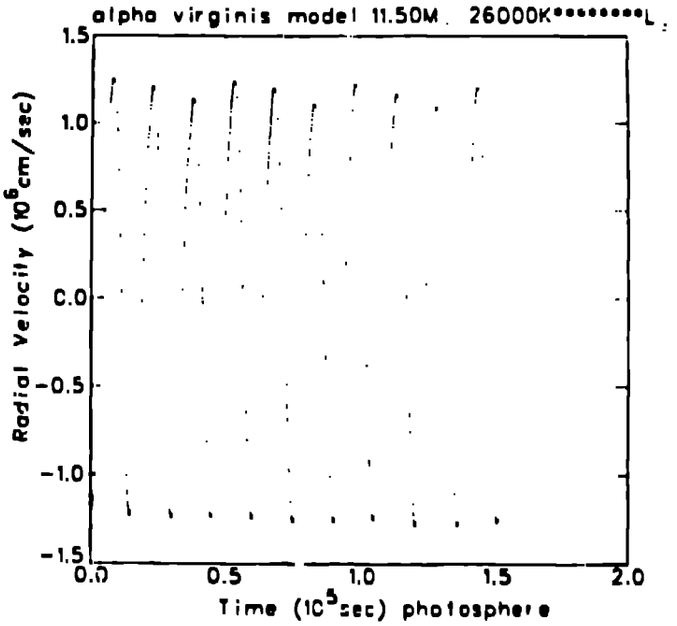

Figure 6. The radial velocity history is shown for the hydrodynamic study of an α Vir model.

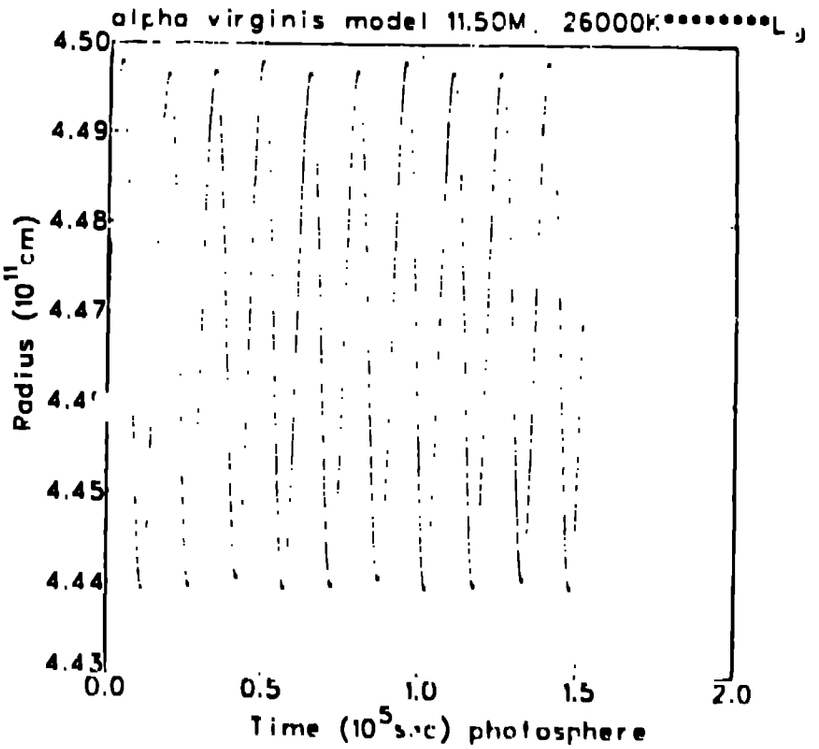

Figure 7. The radius history is shown for the α Vir study.

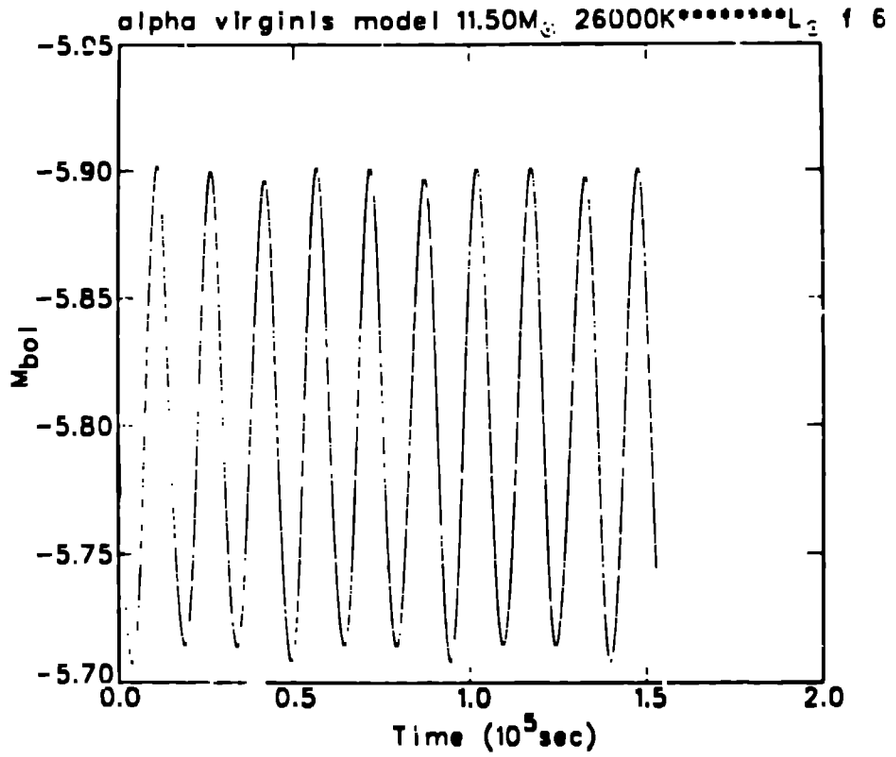

Figure 8. The luminosity history is given for th α Vir study.

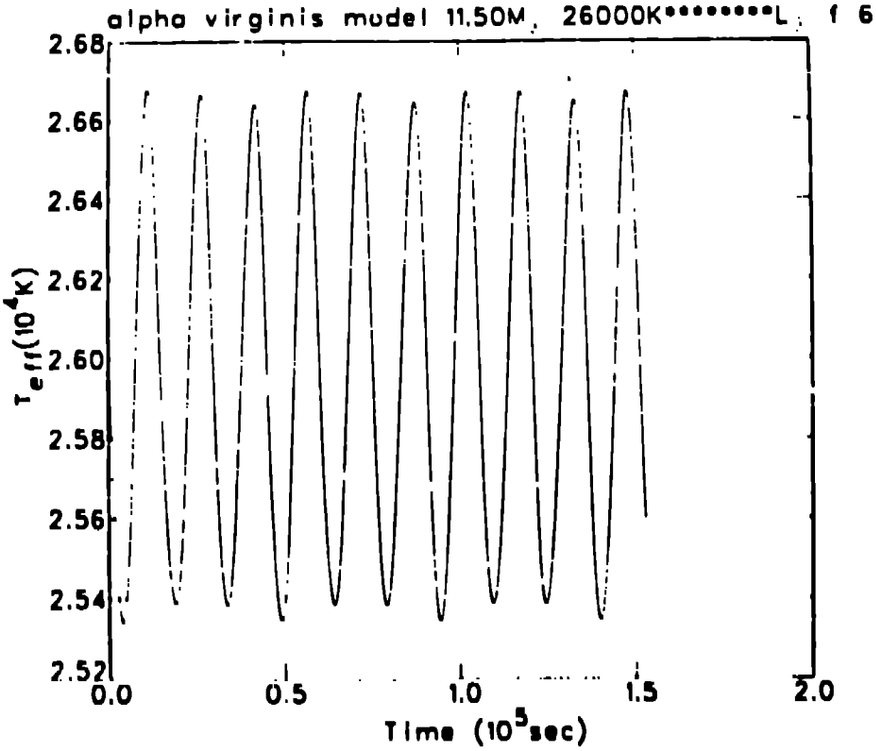

Figure 9. The surface effective temperature history is shown for the α Vir study.

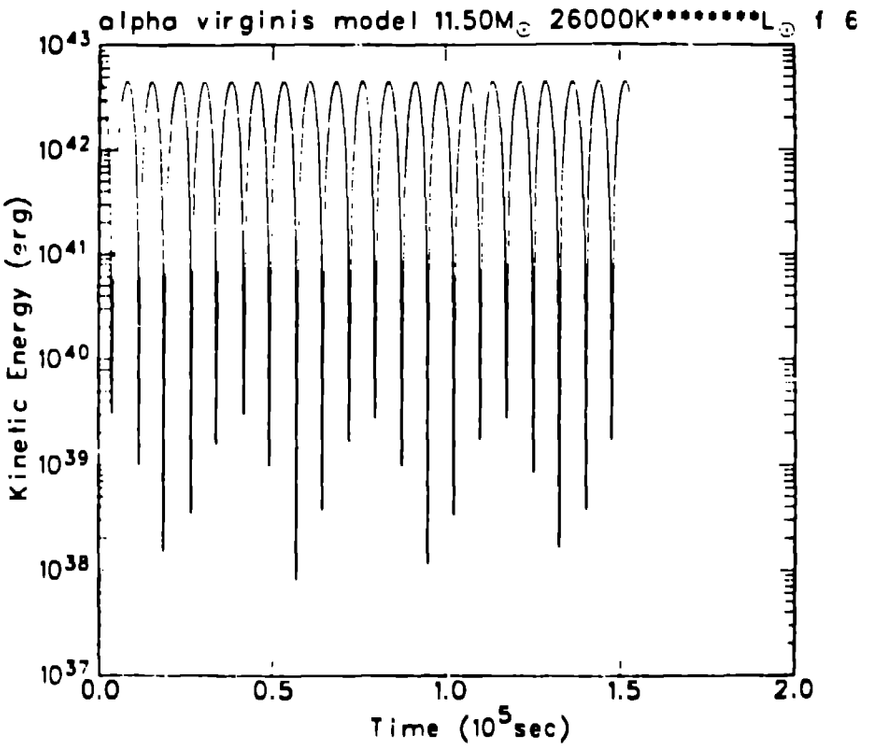

Figure 10. Variations of the total kinetic energy of the pulsating envelope are shown for the α Vir study.

REFERENCES

Appenzeller, I. 1970, Astron. Astrophys. 5, 355.

Baker, N.H. and von Sengbusch, K. 1969, Mitt. der Astron. Gesellschaft, 27, 162.

Bendt, J.E. and Davis, C.G. 1971, Ap. J. 169, 333.

Castor, J.I. 1971, Ap. J. 166, 109.

Castor, J.I., Davis, C.G., and Davison, D.K. 1977, Los Alamos Rept LA-6664.

Christy, R.F. 1964, Rev. Mod. Phys. 36, 555.

Christy, R.F. 1967, Method Comp. Phys. 7 (Academic Press, New York) p191.

Code, A.D., Davis, J., Bless, R. C., and Brown, R. H. 1976, Ap. J. 203, 417.

Cox, A.N., Brownlee, R.R., and Eilers, D.D. 1966, Ap. J. 144, 1024.

Fowler, W.A., Caughlan, G.W., and Zimmerman, B.A. 1975, Ann. Rev. Astron. Astrophys. 13, 69.

Fraley, G.S. 1968, Ast. Space Sci. 2, 96.

Karp, A.H. 1975, Ap. J. 199, 448.

Keeley, D.A. 1970, Ap. J. 161, 657.

Kutter, G.S. and Sparks, W.M. 1972, Ap. J. 175, 407.

Papaloizou, J.C.B. 1973, M.N.R.A.S. 162, 169.

Reeves H. 1965, Stellar Structure, Stars and Stellar Systems (University of Chicago Press) p113.

Richtmyer, R.D. and Morton, K.W. 1967, Difference Methods for Initial Value Problems (Intersciences, New York).

Spangenberg, W.H. 1975, Cepheid Modeling NASA SP-383, p183.

Stellingwerf, R.F. 1974, Ap. J. 192, 139.

Stellingwerf, R.F. 1975, Ap. J. 195, 441.

Stobie, R.S. 1969, M.N.R.A.S. 144, 461.

Talbot, R.J. 1971, Ap. J. 165, 121.

Trimble, V. 1972, M.N.R.A.S. 156, 411.

Vemury, S.K. and Stothers, R. 1978, Ap. J. 225, 999.

Wood, P.R. 1974, ap.J. 190, 609.

Ziebarth, K. 1970, Ap.J. 162, 947.